Topics from harmonic analysis related to generalized Poincaré-Sobolev inequalities: Lecture II

Carlos Pérez

University of the Basque Country
&
BCAM

Summer School on
Dyadic Harmonic Analysis, Martingales, and Paraproducts

Bazaleti, Georgia
September 2-6, 2019

Characterization of pointwise Integral formula

Characterization of pointwise Integral formula

Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\, dx$$

Characterization of pointwise Integral formula

Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

Characterization of pointwise Integral formula

Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

$$|f(x)-f_Q| \le c\, I_1(|\nabla f|\chi_Q)(x) \qquad x\in Q$$

Characterization of pointwise Integral formula

Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

$$|f(x)-f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x\in Q$$

where

$$I_\alpha f(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \qquad 0<\alpha<n$$

Characterization of pointwise Integral formula

Poincaré $(1,1)$ **inequality:**

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

$$|f(x)-f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

where

$$I_\alpha f(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \qquad 0<\alpha<n$$

- These are called **fractional integrals** of order α or **Riesz potentials**

Characterization of pointwise Integral formula

Poincaré $(1,1)$ **inequality:**

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

$$|f(x)-f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

where

$$I_\alpha f(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \qquad 0<\alpha<n$$

- These are called **fractional integrals** of order α or **Riesz potentials**
- This representation formula is key in the area

Characterization of pointwise Integral formula

Poincaré $(1,1)$ **inequality:**

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

It follows from the following pointwise estimate:

$$|f(x)-f_Q| \le c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

where

$$I_\alpha f(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \qquad 0<\alpha<n$$

- These are called **fractional integrals** of order α or **Riesz potentials**
- This representation formula is key in the area
- It goes back to **Sobolev**

Dyadic system in a cube

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n "descendent" from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n "descendent" from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n “descendent” from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

They share the same properties as the usual family of dyadic cubes:

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n “descendent” from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

They share the same properties as the usual family of dyadic cubes:

1) Every two dyadic cubes are either disjoint or one of them contains the other.

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n "descendent" from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

They share the same properties as the usual family of dyadic cubes:

1) Every two dyadic cubes are either disjoint or one of them contains the other.

2) In particular each $x \in Q$ generates a unique one-way inifinite chain $\{Q_k\}_{k=0}^{\infty}$, $Q_{k+1} \subset Q_k$, such that

$$\{x\} = \cap_{k=0}^{\infty} Q_k$$

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n "descendent" from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

They share the same properties as the usual family of dyadic cubes:

1) Every two dyadic cubes are either disjoint or one of them contains the other.

2) In particular each $x \in Q$ generates a unique one-way inifinite chain $\{Q_k\}_{k=0}^{\infty}$, $Q_{k+1} \subset Q_k$, such that

$$\{x\} = \cap_{k=0}^{\infty} Q_k$$

The dyadic construction is intimately to the **dyadic maximal function:**

$$M_Q^d f(x) = M_Q f(x) = \sup_{\substack{P \in \mathcal{D}(Q) \\ x \in P}} \frac{1}{|P|} \int_P |f(y)|\, dy.$$

Dyadic system in a cube

Given a cube Q we can construct its family of dyadic cubes $\mathcal{D}(Q)$

by bisecting each side of the cube Q we obtain 2^n “descendent” from Q which is the first generation D_1; then we keep dividing in this way each descendent.

Then $\quad \mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k \quad$ is the family of dyadic cubes relative to Q.

They share the same properties as the usual family of dyadic cubes:

1) Every two dyadic cubes are either disjoint or one of them contains the other.

2) In particular each $x \in Q$ generates a unique one-way inifinite chain $\{Q_k\}_{k=0}^{\infty}$, $Q_{k+1} \subset Q_k$, such that

$$\{x\} = \cap_{k=0}^{\infty} Q_k$$

The dyadic construction is intimately to the **dyadic maximal function:**

$$M^d_Q f(x) = M_Q f(x) = \sup_{\substack{P \in \mathcal{D}(Q) \\ x \in P}} \frac{1}{|P|} \int_P |f(y)|\, dy.$$

Several characterizations formula

Several characterizations formula

Thm TFAE, Let $g \geq$ (which could be a measure), $0 < \alpha < n$

1) The following Poincaré inequality holds

$$\fint_Q |f(x) - f_Q| dx \leq c\ell(Q)^\alpha \fint_Q g dx,$$

Several characterizations formula

Thm TFAE, Let $g \geq$ (which could be a measure), $0 < \alpha < n$

1) The following Poincaré inequality holds

$$\fint_Q |f(x) - f_Q| dx \leq c\ell(Q)^\alpha \fint_Q g dx,$$

2) The following pointwise estimate holds

$$|f(x) - f_Q| \leq c_n I_\alpha(g\chi_Q)(x) \qquad a.e. x \in Q$$

Several characterizations formula

Thm TFAE, Let $g \geq$ (which could be a measure), $0 < \alpha < n$

1) The following Poincaré inequality holds

$$\fint_Q |f(x) - f_Q| dx \leq c\ell(Q)^\alpha \fint_Q g dx,$$

2) The following pointwise estimate holds

$$|f(x) - f_Q| \leq c_n I_\alpha(g\chi_Q)(x) \qquad a.e. x \in Q$$

3) If μ is any measure on $\mathbb{R}^n$, and $0 < \alpha < n$, then for any cube Q,

$$\|(f - f_Q)\chi_Q\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} \leq C \int_Q g\,(M\mu(x))^{\frac{n-\alpha}{n}},\ dx$$

Several characterizations formula

Thm TFAE, Let $g \geq$ (which could be a measure), $0 < \alpha < n$

1) The following Poincaré inequality holds

$$\fint_Q |f(x) - f_Q| dx \leq c\ell(Q)^\alpha \fint_Q g dx,$$

2) The following pointwise estimate holds

$$|f(x) - f_Q| \leq c_n I_\alpha(g\chi_Q)(x) \qquad a.e. x \in Q$$

3) If μ is any measure on $\mathbb{R}^n$, and $0 < \alpha < n$, then for any cube Q,

$$\|(f - f_Q)\chi_Q\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} \leq C \int_Q g \left(M\mu(x)\right)^{\frac{n-\alpha}{n}}, \ dx$$

We observe that 2) follows directly from 3) by testing with Dirac measures

Corollaries: from local to global

Corollary Let $n \geq 2$ μ is any measure, Q a cube, and if f a Lipschitz function, then

$$\left(\int_Q |f - f_Q|^{n'}\, d\mu\right)^{\frac{1}{n'}} \leq C \int_Q |\nabla f| (M\mu)^{\frac{1}{n'}}\, dx$$

Corollaries: from local to global

Corollary Let $n \geq 2$ μ is any measure, Q a cube, and if f a Lipschitz function, then

$$\left(\int_Q |f - f_Q|^{n'}\, d\mu\right)^{\frac{1}{n'}} \leq C \int_Q |\nabla f| (M\mu)^{\frac{1}{n'}}\, dx$$

Thm **(a weighted isoperimetric)**

Let μ be any measure on $\mathbb{R}^n$, $n \geq 2$. Then there is a dimensional constant c such that for an Lipschitz function f with compact support,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n'}\, d\mu\right)^{\frac{1}{n'}} \leq c \int_{\mathbb{R}^n} |\nabla f(y)| (M\mu(y))^{\frac{1}{n'}}\, dy$$

Corollaries: from local to global

Corollary Let $n \geq 2$ μ is any measure, Q a cube, and if f a Lipschitz function, then

$$\left(\int_Q |f - f_Q|^{n'} \, d\mu\right)^{\frac{1}{n'}} \leq C \int_Q |\nabla f| (M\mu)^{\frac{1}{n'}} \, dx$$

Thm (a weighted isoperimetric)

Let μ be any measure on $\mathbb{R}^n$, $n \geq 2$. Then there is a dimensional constant c such that for an Lipschitz function f with compact support,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n'} \, d\mu\right)^{\frac{1}{n'}} \leq c \int_{\mathbb{R}^n} |\nabla f(y)| (M\mu(y))^{\frac{1}{n'}} \, dy$$

The proof follows from (4) letting $\ell(Q) \to \infty$ and using that $f_Q \to 0$.

Proof I

Proof

1) $\Longrightarrow$ 2). (ideas of a paper by Franchi-Lu-Wheeden) We use the Lebesgue differentiation theorem.

Proof I

Proof

1) $\Longrightarrow$ 2). (ideas of a paper by Franchi-Lu-Wheeden) We use the Lebesgue differentiation theorem.

Let $x \in Q$. Then there is a chain $\{Q_k\}_{k\geq 1}$ of nested *dyadic* subcubes of Q such that $Q_1 = Q$, $Q_{k+1} \subset Q_k$ for all $k \geq 1$ and

$$\{x\} = \bigcap_{k\geq 1} Q_k$$

Proof I

Proof

1) $\Longrightarrow$ 2). (ideas of a paper by Franchi-Lu-Wheeden) We use the Lebesgue differentiation theorem.

Let $x \in Q$. Then there is a chain $\{Q_k\}_{k\geq 1}$ of nested *dyadic* subcubes of Q such that $Q_1 = Q$, $Q_{k+1} \subset Q_k$ for all $k \geq 1$ and

$$\{x\} = \bigcap_{k\geq 1} Q_k$$

Let f_{Q_k} be the average of f over the cube Q_k. Then by the LDD, there exists a null set N such that for all $x \in E := Q \setminus N$

Proof I

Proof

1) $\Longrightarrow$ 2). (ideas of a paper by Franchi-Lu-Wheeden) We use the Lebesgue differentiation theorem.
Let $x \in Q$. Then there is a chain $\{Q_k\}_{k\geq 1}$ of nested *dyadic* subcubes of Q such that $Q_1 = Q$, $Q_{k+1} \subset Q_k$ for all $k \geq 1$ and

$$\{x\} = \bigcap_{k\geq 1} Q_k$$

Let f_{Q_k} be the average of f over the cube Q_k. Then by the LDD, there exists a null set N such that for all $x \in E := Q \setminus N$

$$|f(x) - f_Q| = |\lim_{k\to\infty} f_{Q_k} - f_Q| = |\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}|$$

Proof I

Proof

1) $\Longrightarrow$ 2). (ideas of a paper by Franchi-Lu-Wheeden) We use the Lebesgue differentiation theorem.

Let $x \in Q$. Then there is a chain $\{Q_k\}_{k\geq 1}$ of nested *dyadic* subcubes of Q such that $Q_1 = Q$, $Q_{k+1} \subset Q_k$ for all $k \geq 1$ and

$$\{x\} = \bigcap_{k\geq 1} Q_k$$

Let f_{Q_k} be the average of f over the cube Q_k. Then by the LDD, there exists a null set N such that for all $x \in E := Q \setminus N$

$$|f(x) - f_Q| = |\lim_{k\to\infty} f_{Q_k} - f_Q| = |\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}|$$

Now, using the dyadic structure of the chain, we obtain that

Proof II

Proof II

$$
\begin{aligned}
|\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}| &\leq \sum_k \frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} |f_{Q_k} - f| \\
&\leq 2^n \sum_k \frac{1}{|Q_k|}\int_{Q_k} |f_{Q_k} - f| \\
&\leq C2^n \sum_k \frac{\ell(Q_k)^\alpha}{|Q_k|}\int_{Q_k} g(y)dy \\
&= C2^n \int_Q g(y) \sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)dy
\end{aligned}
$$

Proof II

$$
\begin{aligned}
\Big|\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}\Big| &\leq \sum_k \frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} |f_{Q_k} - f| \\
&\leq 2^n \sum_k \frac{1}{|Q_k|}\int_{Q_k} |f_{Q_k} - f| \\
&\leq C2^n \sum_k \frac{\ell(Q_k)^\alpha}{|Q_k|}\int_{Q_k} g(y)dy \\
&= C2^n \int_Q g(y) \sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)dy
\end{aligned}
$$

To sum up this series as follows.

Proof II

$$\begin{aligned}
\Big|\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}\Big| &\leq \sum_k \frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} |f_{Q_k} - f| \\
&\leq 2^n \sum_k \frac{1}{|Q_k|}\int_{Q_k} |f_{Q_k} - f| \\
&\leq C2^n \sum_k \frac{\ell(Q_k)^\alpha}{|Q_k|}\int_{Q_k} g(y)dy \\
&= C2^n \int_Q g(y) \sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)dy
\end{aligned}$$

To sum up this series as follows. If $0 < \eta < n - \alpha$,

Proof II

$$
\begin{aligned}
\Big|\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}\Big| &\leq \sum_k \frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} |f_{Q_k} - f| \\
&\leq 2^n \sum_k \frac{1}{|Q_k|}\int_{Q_k} |f_{Q_k} - f| \\
&\leq C2^n \sum_k \frac{\ell(Q_k)^\alpha}{|Q_k|}\int_{Q_k} g(y)dy \\
&= C2^n \int_Q g(y)\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)dy
\end{aligned}
$$

To sum up this series as follows. If $0 < \eta < n - \alpha$,

$$
\begin{aligned}
\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) &\leq \frac{C_n}{|x-y|^{n-\alpha-\eta}}\sum_k \ell(Q_k)^{-\eta}\chi_{Q_k}(y) \\
&\leq \frac{C_n}{|x-y|^{n-\alpha-\eta}\ell(Q)^\eta}\sum_{k=0}^{k_0} 2^{k\eta}
\end{aligned}
$$

Proof II

$$
\begin{aligned}
\Big|\sum_{k\geq 1} f_{Q_{k+1}} - f_{Q_k}\Big| &\leq \sum_k \frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} |f_{Q_k} - f| \\
&\leq 2^n \sum_k \frac{1}{|Q_k|}\int_{Q_k} |f_{Q_k} - f| \\
&\leq C2^n \sum_k \frac{\ell(Q_k)^\alpha}{|Q_k|}\int_{Q_k} g(y)dy \\
&= C2^n \int_Q g(y) \sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)dy
\end{aligned}
$$

To sum up this series as follows. If $0 < \eta < n-\alpha$,

$$
\begin{aligned}
\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) &\leq \frac{C_n}{|x-y|^{n-\alpha-\eta}} \sum_k \ell(Q_k)^{-\eta}\chi_{Q_k}(y) \\
&\leq \frac{C_n}{|x-y|^{n-\alpha-\eta}\ell(Q)^\eta} \sum_{k=0}^{k_0} 2^{k\eta}
\end{aligned}
$$

for $k_0 = \min\{j \in \mathbb{N} : 2^j > \sqrt{n}\frac{\ell(Q)}{|x-y|}\}$. Then we obtain that

Proof III

Proof III

$$\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) \le C_n \frac{2^{\eta k_0}}{|x-y|^{n-\alpha-\eta}\ell(Q)^{\eta}} \le C_n \frac{1}{|x-y|^{n-\alpha}}$$

Proof III

$$\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) \leq C_n \frac{2^{\eta k_0}}{|x-y|^{n-\alpha-\eta}\ell(Q)^\eta} \leq C_n \frac{1}{|x-y|^{n-\alpha}}$$

Collecting all previous estimates, we conclude with the desired inequality

$$|f(x) - f_Q| \leq C_n \int_Q \frac{g(y)}{|x-y|^{n-\alpha}} dy \qquad \forall x \in Q$$

Proof III

$$\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) \le C_n \frac{2^{\eta k_0}}{|x-y|^{n-\alpha-\eta}\ell(Q)^\eta} \le C_n \frac{1}{|x-y|^{n-\alpha}}$$

Collecting all previous estimates, we conclude with the desired inequality

$$|f(x) - f_Q| \le C_n \int_Q \frac{g(y)}{|x-y|^{n-\alpha}} dy \qquad \forall x \in Q$$

2) $\Longrightarrow$ 3).

Proof III

$$\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) \le C_n \frac{2^{\eta k_0}}{|x-y|^{n-\alpha-\eta}\ell(Q)^{\eta}} \le C_n \frac{1}{|x-y|^{n-\alpha}}$$

Collecting all previous estimates, we conclude with the desired inequality

$$|f(x) - f_Q| \le C_n \int_Q \frac{g(y)}{|x-y|^{n-\alpha}} dy \qquad \forall x \in Q$$

2) $\Longrightarrow$ 3).

We can compute the weak norm by using normability. More precisely, we can apply Minkowski's inequality to obtain

Proof III

$$\sum_k \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y) \le C_n \frac{2^{\eta k_0}}{|x-y|^{n-\alpha-\eta}\ell(Q)^{\eta}} \le C_n \frac{1}{|x-y|^{n-\alpha}}$$

Collecting all previous estimates, we conclude with the desired inequality

$$|f(x)-f_Q| \le C_n \int_Q \frac{g(y)}{|x-y|^{n-\alpha}}dy \qquad \forall x \in Q$$

2) $\Longrightarrow$ 3).

We can compute the weak norm by using normability. More precisely, we can apply Minkowski's inequality to obtain

$$\begin{aligned}
\|(f-f_Q)\chi_Q\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} &\lesssim \|I_\alpha(g\chi_Q)\chi_Q\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} \\
&\lesssim \|\int_{\mathbb{R}^n} \frac{g(y)\chi_Q(y)}{|\cdot-y|^{n-\alpha}}\chi_Q(\cdot)\, dy\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} \\
&\lesssim \int_{\mathbb{R}^n} \|K(\cdot,y)\chi_Q(\cdot)\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} g(y)\chi_Q(y)\, dy \\
&\lesssim \int_Q \|K(\cdot,y)\chi_Q(\cdot)\|_{L^{\frac{n}{n-\alpha},\infty}(\mu)} g(y)\, dy
\end{aligned}$$

Proof IV

Proof IV

Now we estimate the inner norm of the kernel $K(x,y)=\frac{1}{|x-y|^{n-\alpha}}$, $x,y\in Q$.
By definition of the weak norm, we have

Proof IV

Now we estimate the inner norm of the kernel $K(x,y) = \frac{1}{|x-y|^{n-\alpha}}$, $x, y \in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0}\left(t^{\frac{n}{n-\alpha}}\mu\left(x\in Q : \frac{1}{|x-y|^{n-1}} > t\right)\right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0}\left(r^{-n}\mu\left(x\in Q : |x-y| < r\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0}\left(|B(y,r)|^{-1}\mu\left(B(y,r)\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \left(M^c\mu(y)\right)^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

Proof IV

Now we estimate the inner norm of the kernel $K(x,y) = \frac{1}{|x-y|^{n-\alpha}}$, $x, y \in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0}\left(t^{\frac{n}{n-\alpha}}\mu\left(x\in Q : \frac{1}{|x-y|^{n-1}} > t\right)\right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0}\left(r^{-n}\mu\left(x\in Q : |x-y| < r\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0}\left(|B(y,r)|^{-1}\mu\left(B(y,r)\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \left(M^c\mu(y)\right)^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f-f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} \leq C\int_Q g(y)(M\mu(y))^{\frac{n-\alpha}{n}}\, dy$$

Proof IV

Now we estimate the inner norm of the kernel $K(x,y) = \frac{1}{|x-y|^{n-\alpha}}$, $x, y \in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0} \left(t^{\frac{n}{n-\alpha}} \mu \left(x \in Q : \frac{1}{|x-y|^{n-1}} > t \right) \right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0} \left(r^{-n} \mu \left(x \in Q : |x-y| < r \right) \right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0} \left(|B(y,r)|^{-1} \mu \left(B(y,r) \right) \right)^{\frac{n-\alpha}{n}} \\
&\lesssim \left(M^c \mu(y) \right)^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f - f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} \leq C \int_Q g(y) (M\mu(y))^{\frac{n-\alpha}{n}} \, dy$$

3) $\Longrightarrow$ 4)

Proof IV

Now we estimate the inner norm of the kernel $K(x,y) = \frac{1}{|x-y|^{n-\alpha}}$, $x, y \in Q$.
By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0}\left(t^{\frac{n}{n-\alpha}}\mu\left(x\in Q : \frac{1}{|x-y|^{n-1}} > t\right)\right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0}\left(r^{-n}\mu\left(x\in Q : |x-y| < r\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0}\left(|B(y,r)|^{-1}\mu\left(B(y,r)\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \left(M^c\mu(y)\right)^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f-f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} \le C\int_Q g(y)(M\mu(y))^{\frac{n-\alpha}{n}}\,dy$$

3) $\Longrightarrow$ 4)

This follows from the ”truncation” method (seems to be due to Maz’ja).

Proof IV

Now we estimate the inner norm of the kernel $K(x,y) = \frac{1}{|x-y|^{n-\alpha}}$, $x, y \in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0} \left(t^{\frac{n}{n-\alpha}} \mu \left(x \in Q : \frac{1}{|x-y|^{n-1}} > t \right) \right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0} \left(r^{-n} \mu \left(x \in Q : |x-y| < r \right) \right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0} \left(|B(y,r)|^{-1} \mu \left(B(y,r) \right) \right)^{\frac{n-\alpha}{n}} \\
&\lesssim (M^c \mu(y))^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f - f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} \leq C \int_Q g(y)(M\mu(y))^{\frac{n-\alpha}{n}}\, dy$$

3) $\Longrightarrow$ 4)

This follows from the ”truncation” method (seems to be due to Maz’ja).

4) $\Longrightarrow$ 1).

Proof IV

Now we estimate the inner norm of the kernel $K(x,y)=\frac{1}{|x-y|^{n-\alpha}}$, $x,y\in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0}\left(t^{\frac{n}{n-\alpha}}\mu\left(x\in Q:\frac{1}{|x-y|^{n-1}}>t\right)\right)^{\frac{1}{\frac{n}{n-\alpha}}} \\
&\lesssim \sup_{r>0}\left(r^{-n}\mu\left(x\in Q:|x-y|<r\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \sup_{r>0}\left(|B(y,r)|^{-1}\mu\left(B(y,r)\right)\right)^{\frac{n-\alpha}{n}} \\
&\lesssim \left(M^c\mu(y)\right)^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f-f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}}\le C\int_Q g(y)(M\mu(y))^{\frac{n-\alpha}{n}}\,dy$$

3) $\Longrightarrow$ 4)

This follows from the ”truncation” method (seems to be due to Maz’ja).

4) $\Longrightarrow$ 1).

This follows by considering as measure μ the Lebesgue measure.

Proof IV

Now we estimate the inner norm of the kernel $K(x,y)=\frac{1}{|x-y|^{n-\alpha}}$, $x,y\in Q$. By definition of the weak norm, we have

$$\begin{aligned}
\|K(\cdot,y)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}} &= \sup_{t>0}\left(t^{\frac{n}{n-\alpha}}\mu\left(x\in Q:\frac{1}{|x-y|^{n-1}}>t\right)\right)^{\frac{1}{\frac{n}{n-\alpha}}}\\
&\lesssim \sup_{r>0}\left(r^{-n}\mu\left(x\in Q:|x-y|<r\right)\right)^{\frac{n-\alpha}{n}}\\
&\lesssim \sup_{r>0}\left(|B(y,r)|^{-1}\mu\left(B(y,r)\right)\right)^{\frac{n-\alpha}{n}}\\
&\lesssim (M^c\mu(y))^{\frac{n-\alpha}{n}}
\end{aligned}$$

Recall that M^c denotes the centered maximal function. Therefore, collecting all estimates, we obtain that

$$\|(f-f_Q)\chi_Q\|_{L_\mu^{\frac{n}{n-\alpha},\infty}}\le C\int_Q g(y)(M\mu(y))^{\frac{n-\alpha}{n}}\,dy$$

3) $\Longrightarrow$ 4)

This follows from the ”truncation” method (seems to be due to Maz’ja).

4) $\Longrightarrow$ 1).

This follows by considering as measure μ the Lebesgue measure.

□